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The q-deformation of symmetry transformations preserving the radial form of the Schrödinger equation

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Abstract. Proofs are given that the symmetry transformations preserving the radial form of the $\mathrm{SO}_q(N)$ Schrödinger equation lead to matching conditions which are essentially the same as the classical ones. Power-law potentials are treated as illustrative examples. In particular, the N_1-dimensional q-deformed Coulomb system is converted into a q-deformed harmonic oscillator acting again in $N_2 = 2(N_1 - 1)$ space dimensions. We also found that q-deformed $1/N$ energy formulae are covariant under such transformations to first $1/N$-order.

1. Introduction

The derivation of one-dimensional q-deformed Schrödinger equations has received much interest during the last few years [1–5]. One can proceed by replacing, somewhat tentatively, the usual derivative with the q-difference derivative found long ago by Jackson [6]. The same applies, for example, to the phase-space realization of the q-deformed algebra of oscillators [7]. Therefore, we are faced with q-analysis and q-difference equations. The deformation parameter is denoted by q, so that the usual description is reproduced as $q \to 1$. A further important step has been the synthesis between quantum groups described by R-matrices [8] and the covariant differential calculus [9, 10], such as has been performed on the non-commutative quantum Euclidean space R_q^N [10, 11]. For this purpose, the quantum group of linear matrices $\mathrm{GL}_q(N)$ serves as an illustrative example. The quantum group of rotations $\mathrm{SO}_q(N)$ has also been treated in a similar manner [12–14]. These developments led to the derivation of the $\mathrm{SO}_q(N)$-counterpart of the N-dimensional radial Schrödinger equation, as shown by (2.33) in [15].

Under such circumstances we have to define the equivalent radial Jackson derivative as [16]

$$\partial_q^{(r)} r^n = \frac{\mu}{q+1} [[n]]_q r^{n-1} \tag{1}$$

where $\mu = 1 + q^{2-N}$. Accordingly

$$\partial_q^{(r)} f(r) = \frac{\mu}{q+1} \frac{f(qr) - f(r)}{r(q-1)} \tag{2}$$

where $f(r)$ is an arbitrary analytic function. The rescaled radial differential operator can also be introduced via $\mathrm{d}_q/\mathrm{d}_q r = ((q+1)/\mu)\partial_q^{(r)}$. This time $q > 0$, whereas the quantum number reads

$$[[n]]_q \equiv q^{\frac{n-1}{2}} [n]_{\sqrt{q}} = \frac{q^n - 1}{q-1}. \tag{3}$$

The $\mathrm{SO}_q(N)$-counterpart of the N-dimensional radial Schrödinger equation is then given by

$$\left(-q^{2\ell+N-1}\partial_q^{(r)^2} - \frac{\mu}{q+1}[[2\ell+N-1]]_q\frac{1}{r}\partial_q^{(r)} + V(r)\right)\psi(r) = \mathcal{E}_q\psi(r). \tag{4}$$

Other inter-related radial equations exhibiting non-trivial q-deformations of the centrifugal barrier have also been written down [16]. Relatedly, the q-parameter has a well defined theoretical meaning as it characterizes, in conjunction with the R-matrix, the non-commutative attributes of the underlying quantum Euclidean space. The radial coordinate of this space and the classical quantum number of the angular momentum are denoted by r and ℓ, respectively. The q-deformed energy is $\mathcal{E}_q$ and $\psi(r)$ represents the q-wavefunction. Units for which $\hbar = 2m = 1$ are used.

Now we have to recall that form-preserving canonical transformations of the usual radial Schrödinger equation in N space dimensions have been discussed earlier [17–19]. On the other hand, q-canonical transformations in one space dimension have been discussed recently, with a main emphasis on the rather special one-dimensional oscillator–Coulomb duality [20]. Motivated by these issues, we would like to address the question of whether q-deformed form-preserving canonical transformations can also be performed for the radial $\mathrm{SO}_q(N)$ equation presented above. We shall then see that this is indeed the case, thereby succeeding in generalizing earlier results towards the present q-deformed description.

2. The q-conversion technique

Putting for example, $r = r_2$, let us introduce a new radial coordinate as

$$r_1 = r_1(r_2) = r_2^\alpha \tag{5}$$

where α is a power exponent which remains to be established later. We shall assume that r_1 and r_2 correspond to non-commutative Euclidean spaces having N_1 and N_2 space dimensions, respectively. The quantum numbers of the angular momentum are ℓ_1 and ℓ_2, whereas the corresponding deformation parameters are q_1 and $q_2 = q$. Then the q-wavefunction $\psi_1(r_1)$ yields a transformed q-wavefunction as follows:

$$\psi_2(r_2) = \psi_1(r_1(r_2)) \tag{6}$$

so that

$$\psi_2(q_2r_2) = \psi_1(r_1(q_2r_2)) = \psi_1(q_1r_1). \tag{7}$$

This shows that

$$q_1 = q_2^\alpha = q^\alpha \tag{8}$$

which will be used in the following. We have to recognize that equations (6) and (7) are able to proceed up to a constant factor, but here we shall identify this factor with unity. Under such circumstances, the radial derivative $\partial_{q_2}^{(r_2)}$ becomes subject to the transformation

$$\partial_{q_2}^{(r_2)}\psi_2(r_2) = \partial_{q_2}^{(r_2)}\psi_1(r_1) \tag{9}$$

so that

$$\partial_{q_1}^{(r_1)} = \frac{\mu_1}{q_1+1}\frac{q_2+1}{\mu_2}\left[\left[\frac{1}{\alpha}\right]\right]_{q_1} r_2^{1-\alpha}\partial_{q_2}^{(r_2)} \tag{10}$$

where $\mu_j = 1 + q_j^{2-N_j}$ $(j = 1, 2)$. In addition, q-deformed momentum operators can be defined as

$$p_j = -\mathrm{i}\partial_{q_j}^{(r_j)}. \tag{11}$$

Then the q-deformed commutation relations read

$$p_j r_j - q_j r_j p_j = -\mathrm{i}\frac{\mu_j}{q_j + 1} \tag{12}$$

as it can be easily verified by direct computation. It is understood that the above momentum operators are non-Hermitian, but a subsequent symmetrization can be readily carried out [21].

Next, let us make the rather special choice $N_1 = N_2 = 1$. Then (10) becomes

$$\partial_{q_1}^{(r_1)} = \frac{1}{q+1} r_2^{-1} \partial_q^{(r_2)} \tag{13}$$

if $\alpha = 2$, where $q_2 = q$, $q_1 = q^2$ and $r_1 = r_2^2$. One realizes immediately that (13), as it stands, reproduces (29) in [20]. Moreover, (22) and (23) in [20] are reproduced by (12) under the same conditions. Thus our formulae (10) and (12) accomplish a non-trivial generalization of one-dimensional results which have been presented before.

3. The derivation of q-deformed Schrödinger equations

Now, let us combine (10) with the input q-deformed Schrödinger equation:

$$\left(-q_1^{2\ell_1+N_1-1}\partial_{q_1}^{(r_1)^2} - \frac{\mu_1}{q_1+1}[[2\ell_1 + N_1 - 1]]_{q_1}\frac{1}{r_1}\partial_{q_1}^{(r_1)} + V_1(r_1)\right)\psi_1(r_1) = \mathcal{E}_{q_1}^{(1)}\psi_1(r_1). \tag{14}$$

Keeping in mind that $q_2 = q$, one would then obtain

$$\left(-q^{2\ell_2+N_2-1}\partial_{q}^{(r_2)^2} - \frac{\mu_2}{q+1}[[2\ell_2 + N_2 - 1]]_{q}\frac{1}{r_2}\partial_{q}^{(r_2)} + \Omega r_2^{2\alpha-2}(V_1(r_2^\alpha) - \mathcal{E}_{q_1}^{(1)})\right)\psi_2(r_2) = 0 \tag{15}$$

provided that

$$\frac{\ell_2 + N_2 - 2}{\ell_1 + N_1 - 2} = \alpha. \tag{16}$$

One has

$$\Omega = [[\alpha]]_q^2 \left(\frac{\mu_2(q_1+1)}{\mu_1(q+1)}\right)^2 \tag{17}$$

for which the q-identity

$$\left[\left[\frac{1}{\alpha}\right]\right]_{q_1}^{-1} = [[\alpha]]_q \tag{18}$$

has been used. Furthermore, starting with $\ell_1 = 0$ nothing prevents us from assuming reasonably that $\ell_2 = 0$, in which case (16) exhibits the simplified form

$$\frac{N_2 - 2}{N_1 - 2} = \alpha. \tag{19}$$

This means that the appropriate solution to (16) is given by

$$\frac{N_2 - 2}{N_1 - 2} = \frac{\ell_2}{\ell_1} = \alpha \tag{20}$$

which reproduces identically the classical matching condition (see (15) and/or (18) in [19]). It should be noted that in the classical case the matching condition (20) also characterizes, from the very beginning, the reduced radial equations, i.e. the ones from which the first radial derivative has been removed. Equation (20) should be viewed as a conversion rule which is reminiscent of the same underlying dynamical symmetry (see also [22]), even if the one-to-one correspondence between the states is not preserved. Such a correspondence could be achieved under suitable modifications [23], but here we find it suitable to proceed further without introducing additional parameters. In other words, we found that under the nonlinear coordinate transformation (5), the number of space dimensions, as well as the quantum number of the angular momentum, become subject to the same inter-related transformations as in the classical case. In general, the classical SO(N) description is meaningful in so far as $N_j > 2$, but one has the possibility of approaching $N_j = 2$ from the right. Extrapolations such as $N_1 = 1$ and/or $N_2 = 1$ are special cases which should be handled with care. It is also understood that the radial quantum number $n_{\mathrm{r}} = 0, 1, 2, \ldots$ is not affected by such transformations. Applying (20) then gives $\mu_1 = \mu_2 = \mu$, which represents a sensible simplification. Accordingly, Ω becomes

$$\Omega = \Omega_0(\alpha) = [[\alpha]]_q^2 \left(\frac{q^\alpha + 1}{q + 1}\right)^2 \tag{21}$$

so that $\Omega_0 \to \alpha^2$ as $q \to 1$.

4. Concrete examples

Let us consider the input power-law potential

$$V_1(r_1) = \frac{C_{n_1}^{(1)}}{r_1^{n_1}}. \tag{22}$$

Then (15) yields a q-deformed radial Schrödinger equation reproducing the form of (4) as soon as

$$\alpha = \alpha_0 = \frac{2}{2 - n_1} \tag{23}$$

which expresses the α-fixing anticipated above. We then find that (15) and (23) imply a duality transformation in which couplings get converted into energies and *vice versa*. This conversion is characterized by the mutual relationships

$$\mathcal{E}_{q_2}^{(2)}(n_2) = -\Omega_0(\alpha_0) C_{n_1}^{(1)} \tag{24}$$

and

$$\Omega_0(\alpha_0)\mathcal{E}_{q_1}^{(1)}(n_1) = -C_{n_2}^{(2)} \tag{25}$$

where

$$n_2 = \frac{2n_1}{n_1 - 2} \tag{26}$$

and where

$$V_2(r_2) = \frac{C_{n_2}^{(2)}}{r_2^{n_2}} \tag{27}$$

stands for the output potential. The q-deformed energies of the input and output systems have been denoted by $\mathcal{E}_{q_1}^{(1)}(n_1)$ and $\mathcal{E}_{q_2}^{(2)}(n_2)$, respectively.

In particular, $\alpha_0 = 2$ if $n_1 = 1$, so that the input Coulomb system $(V_1(r_1) = -Z/r_1)$ in N_1 space dimensions is converted into an harmonic oscillator working in

$$N_2 = 2(N_1 - 1) \tag{28}$$

space dimensions. Correspondingly $V_2(r_2) = \omega^2 r_2^2$, such that

$$\mathcal{E}_q^{(2)}(-2) = Z\Omega_0(2) \tag{29}$$

and

$$\omega^2 = -\Omega_0(2)\mathcal{E}_{q^2}^{(1)}(1) \tag{30}$$

where $\Omega_0(2) = (1 + q^2)^2$. On the other hand, the actual q-deformed energies of the Coulomb- and harmonic-oscillator systems are given by [14, 16]

$$\mathcal{E}_{q^2}^{(1)}(1) = -\frac{(1+q^2)^2}{\mu^2} \frac{Z^2 q^{4d_1}}{[[2d_1]]_{q^2}^2} \tag{31}$$

and [13, 15]

$$\mathcal{E}_q^{(2)}(-2) = \frac{\mu\omega}{q+1} \frac{[[2d_2]]_q}{q^{d_2}} \tag{32}$$

respectively, where

$$d_1 = \ell_1 + n_r + \frac{N_1 - 1}{2} \tag{33}$$

and

$$d_2 = \ell_2 + 2n_r + \frac{N_2}{2} = 2d_1. \tag{34}$$

It is obvious that (29) is fulfilled precisely via (30)–(32), which illustrates in more detail the conversion referred to previously.

Now, generalizations to arbitrary attractive $n < 2$ potentials are in order. For this purpose we shall resort to the $\mathrm{SO}_q(N)$ motivated deformation of $1/N$ energy formulae proposed recently [16]. Using (29)–(31) in [16], one finds that the q-deformed energy of (22) is given by

$$\mathcal{E}_{q_1}^{(1)}(n_1) = \frac{n_1 - 2}{n_1} \left(-\frac{n_1}{2} C_{n_1}^{(1)}\right)^{\frac{2}{2-n_1}} d_{q_1}^{(1)}(n_1)^{\frac{2n_1}{n_1-2}} \tag{35}$$

where $q_1 = q^{\alpha_0}$ and (see also [24])

$$d_{q_1}^{(1)}(n_1) = \frac{\mu}{2q_1} \left[\ell_1 + \frac{N_1 - 2}{2} + \left(n_r + \frac{1}{2}\right)\sqrt{2 - n_1}\right]_{q_1} \tag{36}$$

to first $1/N$-order. The potential (27) can be treated in a similar manner, which results in the q-deformed energy $\mathcal{E}_q^{(2)}(n_2)$. Invoking (23) again, it is an easy matter to verify that (24) and (25) are fulfilled, this time to the first $1/N$-order mentioned above. This is a non-trivial finding which sheds a new light on the interplay between q-deformations and the quantum-mechanical $1/N$ description.

It should also be mentioned that the q-wavefunctions characterizing the inter-related radial equations (35) and (36) in [16] are

$$\Phi_1(r_1) = r_1^{\ell_1}\psi_1(r_1) = \Phi_2(r_2) = r_2^{\ell_2}\psi_2(r_2) \tag{37}$$

and

$$\tilde{\Phi}_1(r_1) = r_1^{-a}\psi_1(r_1) = \tilde{\Phi}_2(r_2) = r_2^{-\alpha_0 a}\psi_2(r_2) \tag{38}$$

respectively, where

$$a = \frac{1}{\alpha_0 \ln q} \ln \frac{1 + q^{-2\ell_2 - N_2 + 2}}{1 + q^{\alpha_0}}. \tag{39}$$

In addition, the matching condition (20) is preserved. It is obvious that (37) and (38) reflect the conversion of q-Laguerre-functions into q-Hermite ones and *vice versa*. Moreover, (20) remains invariant under the modified $\mathrm{SO}_q(N)$ equations presented previously (see (16) and (24) in [16]).

5. Conclusions

Proofs have been given that the symmetry transformations preserving the radial form of the $\mathrm{SO}_q(N)$ Schrödinger equation (4) proceed in close agreement with the classical description. Considering power-law potentials, we have then found the matching condition (20), which reproduces precisely the classical result. The mutual conversion of couplings into energies is expressed by (24) and (25). These equations can be gathered together as

$$\frac{\mathcal{E}_q^{(2)}(n_2)}{C_{n_1}^{(1)}} = \frac{C_{n_2}^{(2)}}{\mathcal{E}_{q_1}^{(1)}(n_1)} = -\Omega_0(\alpha_0) \tag{40}$$

which makes clear the symmetry attributes of the present form-preserving transformations. It has also been found that q-deformed $1/N$ energy formulae fulfil (40) to first $1/N$-order. This agreement supports in turn the reliability of such $1/N$ formulae. Other input potentials can be treated in a similar manner, provided that one succeeds in separating a constant term standing for $-\mathcal{E}_q^{(2)}/\Omega_0(\alpha)$ from the α-dependent product

$$P(\alpha, r_2) = r_2^{2\alpha-2} V_1(r_2^{\alpha}). \tag{41}$$

This leads to a corresponding generalization of (23).

Concerning the mutual relationships discussed above, we have to say that a non-trivial q-deformed energy for the Coulomb system can also be established specifically in terms of a more general deformation of the harmonic oscillator [25]. Moreover, an alternative q-deformation of the energy of the three-dimensional Coulomb system has been obtained from the $\mathrm{SU}_q(2)$-deformation of a four-dimensional oscillator [26], by invoking the Kustaanheimo–Stiefel transformation [27]. It is also well known that the radial N-dimensional Schrödinger equation exhibits the SO(2,1)-symmetry [28–30]. This indicates that further progress can be done by applying the quantum group $\mathrm{SO}_q(2,1)$. Previous results concerning $\mathrm{SU}_q(1,1)$ [7, 31] can then be used by virtue of the $\mathrm{SU}_q(1,1) \sim \mathrm{SO}_q(2,1)$-isomorphism. Proceeding in this way q-deformed ladder and shift operators have been established for the radial-Coulomb, radial-harmonic-oscillator and one-dimensional Morse-oscillator potentials [32]. Moreover, the $\mathrm{SO}_q(2,1)$-deformation of the Morse-oscillator has already been carried out [33]. In this context the q-deformation of the mapping of the radial Coulomb system into the one-dimensional Morse-oscillator deserves further attention. It is then clear that other developments should also concern the $\mathrm{SO}_q(2,1)$-deformation of the radial Schrödinger equation.

Strictly speaking, the one-dimensional q-canonical transformations discussed previously [20] are not oscillator–Coulomb duality transformations, in the sense that the oscillator potential is mapped into a Coulomb potential acting on the positive half-axis, which is supplemented by an attractive inverse square potential. Such inverse square potentials arise typically when one keeps invariant the number of space dimensions [34, 35]. In addition, the one-dimensional limit of the Coulomb potential is $-Z/|x|$, where $x \in (-\infty, +\infty)$ [36]. Accordingly, one dimensional studies are quite desirable. Indeed, the one-dimensional limit of the Coulomb energy is not obtainable automatically from the N-dimensional result by just putting $N = 1$ so that a special treatment is in order. Note, however, that (38) in [20] is misleading, which means that related refinements are in order.

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